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A Pair of Two-Step Sixth Stage Implicit Runge-Kutta Type Methods for First Order Ordinary Differential Equations

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Abstract

We use the multistep collocation technique to derive a pair of two-step block hybrid methods incorporating two off-step interpolation points and one off-step collocation point. The block hybrid methods are then reformulated into a pair of new sixth-stage implicit Runge-Kutta-type methods, which are implemented in solving first-order ordinary differential equations. Apart from establishing the basic convergence of the new methods, the new methods are A-stable. Numerical results indicate that the methods compete favorably with similar existing methods in the literature and should be preferred for these types of problems.

Keywords: Block hybrid method; implicit Runge-Kutta, interpolation, collocation, Butcher tableau.

Introduction

Ordinary differential equations (ODEs) arise frequently in the study of the physical world, and most of their solutions cannot be expressed in closed form using standard methods of calculus. This is why the ability to obtain accurate numerical solutions is important. This research focuses on the first-order initial value problem (IVP) for ODEs of the form

$$y' = f(x, y), y(a) = y_0 \qquad (1)$$

on the interval $a \le x \le b$ where the function $f(x, y) : \mathrm{R} \times \mathrm{R}^m \to \mathrm{R}^m$ is continuous in x and further assume the existence and uniqueness of the solution for (1).

Traditionally, researchers have used one of the two classes of methods for obtaining numerical solutions to (1), which are Runge-Kutta (RK) methods and linear multistep methods (LMM) [12]. Some major disadvantages of the LMM approach are its poor stability as the step number increases with accuracy and the need for additional starting values with a constant step size, sometimes using other one-step methods. For the A-stability property, which is desirable for stiff problems, the order of the LMM is limited to two by Dahlquist barriers [14].
On the other hand, RK methods enjoy higher popularity as opposed to the LMM because of their symmetrical forms, simple coefficients, efficiency, and numerical stability [1]. The RK methods are mainly single-step methods and do not require successive high derivatives of a function; therefore, they are general-purpose initial value problem solvers. The most popular RK method is the classical RK method of order four. It is symmetrical in form and has simple coefficients. The method is well suited for computers because it needs no special starting procedure, makes light demands on storage, repeatedly uses the same straightforward computational procedure, and is numerically stable [11]. RK methods are always referred to as the best-known one-step methods [14]. Furthermore, the methods are fairly simple to program and easy to implement, and their truncation error can be controlled in a more straightforward manner than LMM [7]. The importance and wide applicability of RK methods for both ODEs and partial differential equations (PDEs) have continued to attract the attention of researchers (see, [4], [6].
Whereas [11] derived a sixth-stage implicit RK by reformulating a three-step block hybrid method, in this research, we obtain a pair of new sixth-stage implicit RK types that are reformulated from two-step block hybrid methods obtained via the multistep collocation (MC) technique. This reduced step number and the particular choice of interpolation and collocation points have produced more efficient and accurate methods, as this research has indicated.

Materials and Methods

We developed a pair of new two sixth-stage implicit Runge-Kutta type methods (TSIRK). A continuous formulation of $r > 0$ interpolation points and $m > 0$ collocation points and $0 < m < k+1$ takes the form:

$$y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \beta_j(x) f\left(x_j, y(x_j)\right) \tag{2}$$

where $\alpha_j(x), j = 0(1)r-1,\ \beta_j(x), j = 0(1)m$ are the continuous coefficients of the discrete scheme, $\overline{y}_{n+j}$ are approximations to y_{n+j} with

$$\begin{aligned} y_{n+j} &= y(x_{n+j}) \\ y'(x_{n+j}) &= f_{n+j} \end{aligned} \tag{3}$$

and h is assumed to be the constant step size with $h := x_{n+1} - x_n, n = 0,1,\ldots,N,\ \ hN = b-a$ on a set of spaced points of integration also given by $a = x_0 < x_1 < \ldots < x_n < x_{n+1} < \ldots < x_{n+k} \ldots < x_N = b$.

Set $p = r + m$, hence (2) is of degree $(p-1)$ where $\alpha_j(x)$ and $\beta_j(x)$ are assumed polynomials of the form

$$\begin{aligned} \alpha_j(x) &= \sum_{i=0}^{p-1} \alpha_{j,i+1} x^i, j \in \{0,1,\ldots,r-1\} \\ h\beta_j(x) &= h \sum_{i=0}^{p-1} \beta_{j,i+1} x^i, j = 0(1)m-1 \end{aligned} \tag{4}$$

where the constants $\alpha_{j,i+1}$ and $\beta_{j,i+1}$ are to be determined. Substituting (3) into (2) yields the continuous scheme

$$y(x) = \sum_{j=0}^{r-1} \sum_{i=0}^{p-1} \alpha_{j,i+1} x^i y_{n+j} + h \sum_{j=0}^{m-1} \sum_{i=0}^{p-1} \beta_{j,i+1} x^i f_{n+j} \tag{5}$$

which simplifies to

$$y(x) = \sum_{i=0}^{p-1} \left(\sum_{j=0}^{r-1} \alpha_{j,i+1} y_{n+j} + h \sum_{j=0}^{m-1} \beta_{j,i+1} f_{n+j} \right) x^i = \sum_{i=0}^{p-1} \xi_i x^i \tag{6}$$

where

$$\xi_i = \sum_{j=0}^{r-1} \alpha_{j,i+1} y_{n+j} + h \sum_{j=0}^{m-1} \beta_{j,i+1} f_{n+j} \tag{7}$$

From (6),

$$y'(x) = \sum_{i=0}^{p-1} i \xi_i x^{i-1} \tag{8}$$

In this research, a pair of two-steps $(k = 2)$ continuous hybrid schemes lead to two cases:

Case I: $r = 4$ interpolation points $x_{n+r}, r = 0, \frac{1}{3}, \frac{2}{3}, 1$ in (6) and $m = 3$ collocation points $x_{n+m}, m = 1, \frac{3}{2}, 2$ in (8).

The matrix equation is of the form

$$V\xi = y \tag{9}$$

with V (being of dimension $(r+m)\times(r+m)$) taking the form

$$V=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+\frac{1}{3}} & x_{n+\frac{1}{3}}^2 & x_{n+\frac{1}{3}}^3 & x_{n+\frac{1}{3}}^4 & x_{n+\frac{1}{3}}^5 & x_{n+\frac{1}{3}}^6 \\ 1 & x_{n+\frac{2}{3}} & x_{n+\frac{2}{3}}^2 & x_{n+\frac{2}{3}}^3 & x_{n+\frac{2}{3}}^4 & x_{n+\frac{2}{3}}^5 & x_{n+\frac{2}{3}}^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 \\ 0 & 1 & 2x_{n+\frac{3}{2}} & 3x_{n+\frac{3}{2}}^2 & 4x_{n+\frac{3}{2}}^3 & 5x_{n+\frac{3}{2}}^4 & 6x_{n+\frac{3}{2}}^5 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 \end{bmatrix}$$

and

$$\xi = \left(\xi_0, \xi_1, \ldots, \xi_6\right)^T ; y = \left(y_n, y_{n+\frac{1}{3}}, y_{n+\frac{2}{3}}, y_{n+1}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}\right).$$

To solve for ξ's in (9), we use Maple 18 software and substitute the result into (6) with $t = \dfrac{x - x_n}{h}$ to obtain a two-step continuous hybrid method of the form

$$y(t) = \sum_{j=0}^{k+1} \alpha_{\frac{j}{3}}(t)\, y_{n+\frac{j}{3}} + h\left(\sum_{j=1}^{k} \beta_j(t)\, f_{n+j} + \beta_{\frac{3}{2}}(t)\, f_{n+\frac{3}{2}}\right) \tag{10}$$

where:

$$\alpha_0(t) = \tfrac{26991}{22144}t^6 - \tfrac{185319}{22144}t^5 + \tfrac{499657}{22144}t^4 - \tfrac{674157}{22144}t^3 + \tfrac{29863}{1384}t^2 - \tfrac{41781}{5536}t + 1$$

$$\alpha_{\frac{1}{3}}(t) = -\tfrac{177147}{22144}t^6 + \tfrac{1158867}{22144}t^5 - \tfrac{2901501}{22144}t^4 + \tfrac{3473361}{22144}t^3 - \tfrac{124497}{1384}t^2 + \tfrac{109593}{5536}t$$

$$\alpha_{\frac{2}{3}}(t) = \tfrac{560601}{22144}t^6 - \tfrac{3490209}{22144}t^5 + \tfrac{8128431}{22144}t^4 - \tfrac{8729451}{22144}t^3 + \tfrac{264465}{1384}t^2 - \tfrac{175203}{5536}t$$

$$\alpha_1(t) = -\tfrac{410445}{22144}t^6 + \tfrac{2516661}{22144}t^5 - \tfrac{5726587}{22144}t^4 + \tfrac{5930247}{22144}t^3 - \tfrac{169831}{1384}t^2 + \tfrac{107391}{5536}t$$

$$\beta_1(t) = -\tfrac{54213}{11072}t^6 - \tfrac{322029}{11072}t^5 + \tfrac{2108041}{33216}t^4 - \tfrac{2074637}{33216}t^3 + \tfrac{56845}{2076}t^2 - \tfrac{11623}{2768}t$$

$$\beta_{\frac{3}{2}}(t) = -\tfrac{231}{346}t^6 + \tfrac{1215}{346}t^5 - \tfrac{6931}{1038}t^4 + \tfrac{6047}{1038}t^3 - \tfrac{1208}{519}t^2 + \tfrac{58}{173}t$$

Evaluating (10) at the points $t = \frac{3}{2}, 2$ and its first derivative at the points $t = 0, \frac{1}{3}, \frac{2}{3}$ respectively yields the following five hybrid schemes

$$y_{n+\frac{3}{2}} + \frac{650475}{1417216}y_{n+1} - \frac{2536191}{1417216}y_{n+\frac{2}{3}} + \frac{528525}{1417216}y_{n+\frac{1}{3}} - \frac{60025}{1417216}y_n = h\left(\frac{503475}{708608}f_{n+1} + \frac{4095}{22144}f_{n+\frac{3}{2}} - \frac{3675}{708608}f_{n+1}\right)$$

$$y_{n+2}-\frac{2225}{692}y_{n+\frac{3}{5}}+\frac{2025}{692}y_{n+\frac{2}{3}}-\frac{567}{692}y_{n+\frac{1}{3}}+\frac{75}{692}y_n=h\left(-\frac{475}{1038}f_{n+1}+\frac{400}{519}f_{n+\frac{3}{2}}+\frac{155}{1038}f_{n+2}\right)$$

$$107391y_{n+1}-175203y_{n+\frac{2}{5}}+109593y_{n+\frac{1}{3}}-37870y_n=h\left(5536f_n+23246f_{n+1}-1856f_{n+\frac{3}{2}}+194f_{n+2}\right) \quad (11)$$

$$4312035y_{n+\frac{2}{3}}-2194815y_{n+1}-1913625y_{n+\frac{1}{3}}-203595y_n=h\left(448416f_{n+\frac{1}{3}}-432110f_{n+1}+28480f_{n+\frac{3}{2}}-2786f_{n+2}\right)$$

$$222045y_{n+1}-173745y_{n+\frac{2}{3}}-52245y_{n+\frac{1}{3}}+3945y_n=h\left(56052f_{n+\frac{2}{3}}+34090f_{n+1}-1472f_{n+\frac{3}{2}}+130f_{n+1}\right)$$

Writing (11) in the normalized block form gives

$$A_0Y_s=A_1Y_{s-1}+h\left(B_0F_{s-1}+B_1F_s\right) \quad (12)$$

where s represents the block number, $Y_s=\left(y_{n+\frac{1}{3}},y_{n+\frac{2}{3}},y_{n+1},y_{n+\frac{3}{2}},y_{n+2}\right)$, $Y_{s-1}=\left(y_{n-\frac{5}{3}},y_{n-\frac{4}{3}},y_{n-1},y_{n-\frac{1}{2}},y_n\right)$, $F_s=\left(f_{n+\frac{1}{3}},f_{n+\frac{2}{3}},f_{n+1},f_{n+\frac{3}{2}},f_{n+2}\right)$ and $F_{s-1}=\left(f_{n-\frac{5}{3}},f_{n-\frac{4}{3}},f_{n-1},f_{n-\frac{1}{2}},f_n\right)$,

$$A_0=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},A_1=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{pmatrix},B_0=\begin{pmatrix}0&0&0&0&\frac{3263}{29160}\\0&0&0&0&\frac{382}{3645}\\0&0&0&0&\frac{13}{120}\\0&0&0&0&\frac{123}{1280}\\0&0&0&0&\frac{2}{15}\end{pmatrix} \text{ and}$$

$$B_1=\begin{pmatrix}\frac{4013}{12600}&-\frac{139}{900}&\frac{671}{9720}&-\frac{1688}{127575}&\frac{77}{48600}\\\frac{106}{225}&\frac{31}{450}&\frac{34}{1215}&-\frac{128}{18225}&\frac{11}{12150}\\\frac{621}{1400}&\frac{27}{100}&\frac{23}{120}&-\frac{8}{525}&\frac{1}{600}\\\frac{729}{1400}&\frac{729}{12800}&\frac{207}{320}&\frac{129}{700}&-\frac{63}{12800}\\\frac{54}{175}&\frac{27}{50}&\frac{2}{15}&\frac{128}{175}&\frac{23}{150}\end{pmatrix} \text{ are } 5\times5 \text{ matrices.}$$

We represent (12) using A_0,A_1,B_0 and B_1 in the Butcher tableau of the form

$$\begin{array}{c|c} c & A \\ \hline & b \end{array} \quad (13)$$

where $A=a_{i,j}, b=b_i$ and $c=c_i\left(i=1,2,\ldots,s\right)=\sum_{j=1}^{s}a_{i,j}$ = $\sum_{j=1}^{s}a_{i,j}$ yields

$$\begin{array}{c|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & \frac{3263}{58320} & \frac{4013}{25200} & -\frac{139}{1800} & \frac{671}{19440} & -\frac{844}{127575} & \frac{77}{97200} \\
\frac{1}{3} & \frac{191}{3645} & \frac{53}{225} & \frac{31}{900} & \frac{17}{1215} & -\frac{64}{18225} & \frac{11}{24300} \\
\frac{1}{2} & \frac{13}{240} & \frac{621}{2800} & \frac{27}{200} & \frac{23}{240} & -\frac{4}{525} & \frac{1}{1200} \\
\frac{3}{4} & \frac{123}{2560} & \frac{729}{2800} & \frac{729}{25600} & \frac{207}{640} & \frac{129}{1400} & -\frac{63}{25600} \\
1 & \frac{1}{15} & \frac{27}{175} & \frac{27}{100} & \frac{1}{15} & \frac{64}{175} & \frac{23}{300} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \frac{1}{15} & \frac{27}{175} & \frac{27}{100} & \frac{1}{15} & \frac{64}{175} & \frac{23}{300}
\end{array} \tag{14}$$

Expressing (14) as an implicit Runge-Kutta type method yields

$$y_{n+1} = y_n + h\left(\frac{1}{15}k_1 + \frac{27}{175}k_2 + \frac{27}{100}k_3 + \frac{1}{15}k_4 + \frac{64}{175}k_5 + \frac{23}{300}k_6\right) \tag{15}$$

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{1}{6}h, y_n + h\left(\frac{3263}{58320}k_1 + \frac{4013}{25200}k_2 - \frac{139}{1800}k_3 + \frac{671}{19440}k_4 - \frac{844}{127575}k_5 + \frac{77}{97200}k_6\right)\right)$$

$$k_3 = f\left(x_n + \frac{1}{3}h, y_n + h\left(\frac{191}{3645}k_1 + \frac{53}{225}k_2 + \frac{31}{900}k_3 + \frac{17}{1215}k_4 - \frac{64}{18225}k_5 + \frac{11}{24300}k_6\right)\right)$$

$$k_4 = f\left(x_n + \frac{1}{2}h, y_n + h\left(\frac{13}{240}k_1 + \frac{621}{2800}k_2 + \frac{27}{200}k_3 + \frac{23}{2400}k_4 - \frac{4}{525}k_5 + \frac{1}{1200}k_6\right)\right)$$

$$k_5 = f\left(x_n + \frac{3}{4}h, y_n + h\left(\frac{123}{2560}k_1 + \frac{729}{2800}k_2 + \frac{729}{25600}k_3 + \frac{207}{640}k_4 + \frac{129}{1400}k_5 - \frac{63}{25600}k_6\right)\right)$$

$$k_6 = f\left(x_n + h, y_n + h\left(\frac{1}{15}k_1 + \frac{27}{175}k_2 + \frac{27}{100}k_3 + \frac{1}{15}k_4 + \frac{64}{175}k_5 + \frac{23}{300}k_6\right)\right)$$

The method (15) is the first two-step sixth stage implicit Runge -Kutta type method code named TSIRK1 in this research.

Case II: $r = 4$ interpolation points at $x_{n+r}, r = 0, \frac{1}{4}, \frac{3}{4}, 1$ in (6) and $m = 3$ collocation points at $x_{n+m}, m = 1, \frac{3}{2}, 2$ in (8). Similarly, repeating the procedure in case I yields the normalized block form (12), where in this case,

$Y_s = \left(y_{n+\frac{1}{4}}, y_{n+\frac{3}{4}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}\right)$, $Y_{s-1} = \left(y_{n-\frac{5}{3}}, y_{n-\frac{4}{3}}, y_{n-1}, y_{n-\frac{1}{2}}, y_n\right)$,

$F_s = \left(f_{n+\frac{1}{3}}, f_{n+\frac{2}{3}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}\right)$ and $F_{s-1} = \left(f_{n-\frac{5}{3}}, f_{n-\frac{4}{3}}, f_{n-1}, f_{n-\frac{1}{2}}, f_n\right)$ with A_0 and A_1 same as in case I but

$$B_0=\begin{pmatrix}0&0&0&0&\frac{1441}{15360}\\0&0&0&0&\frac{331}{5120}\\0&0&0&0&\frac{1}{15}\\0&0&0&0&\frac{17}{320}\\0&0&0&0&\frac{13}{135}\end{pmatrix} \text{ and } B_1=\begin{pmatrix}\frac{583}{3150}&-\frac{121}{1800}&\frac{1039}{23040}&-\frac{221}{28800}&\frac{487}{537600}\\\frac{297}{700}&\frac{73}{200}&-\frac{297}{2560}&\frac{43}{3200}&-\frac{243}{179200}\\\frac{656}{1575}&\frac{112}{225}&\frac{1}{90}&\frac{2}{225}&-\frac{1}{1050}\\\frac{81}{175}&\frac{7}{25}&\frac{81}{160}&\frac{41}{200}&-\frac{81}{11200}\\\frac{512}{1575}&\frac{512}{675}&-\frac{4}{45}&\frac{512}{675}&\frac{79}{525}\end{pmatrix}.$$

The equivalent Butcher tableau (13) for case II yields

$$\begin{array}{c|cccccc}0&0&0&0&0&0&0\\\frac{1}{8}&\frac{1441}{30720}&\frac{583}{6300}&-\frac{121}{3600}&\frac{1039}{46080}&-\frac{221}{57600}&\frac{487}{1075200}\\\frac{3}{8}&\frac{331}{10240}&\frac{297}{1400}&\frac{73}{400}&-\frac{297}{5120}&\frac{43}{6400}&-\frac{243}{358400}\\\frac{1}{2}&\frac{1}{30}&\frac{328}{1575}&\frac{56}{225}&\frac{1}{180}&\frac{1}{225}&-\frac{1}{2100}\\\frac{3}{4}&\frac{17}{640}&\frac{81}{350}&\frac{7}{50}&\frac{81}{320}&\frac{41}{400}&-\frac{81}{22400}\\1&\frac{13}{270}&\frac{256}{1575}&\frac{256}{675}&-\frac{2}{45}&\frac{256}{675}&\frac{79}{1050}\\\hline 1&\frac{13}{270}&\frac{256}{1575}&\frac{256}{675}&-\frac{2}{45}&\frac{256}{675}&\frac{79}{1050}\end{array} \tag{16}$$

Similarly, expressing (16) as another implicit Runge-Kutta type method yields

$$y_{n+1}=y_n+h\left(\frac{13}{270}k_1+\frac{256}{1575}k_2+\frac{256}{675}k_3-\frac{2}{45}k_4+\frac{256}{675}k_5+\frac{79}{1050}k_6\right) \tag{17}$$

where

$$k_1=f\left(x_n,y_n\right)$$

$$k_2=f\left(x_n+\frac{1}{8}h,y_n+h\left(\frac{1441}{30720}k_1+\frac{583}{6300}k_2-\frac{121}{3600}k_3+\frac{1039}{46080}k_4-\frac{221}{57600}k_5+\frac{487}{1075200}k_6\right)\right)$$

$$k_3=f\left(x_n+\frac{3}{8}h,y_n+h\left(\frac{331}{10240}k_1+\frac{297}{1400}k_2+\frac{73}{400}k_3-\frac{297}{5120}k_4+\frac{43}{6400}k_5-\frac{243}{358400}k_6\right)\right)$$

$$k_4=f\left(x_n+\frac{1}{2}h,y_n+h\left(\frac{1}{30}k_1+\frac{328}{1575}k_2+\frac{56}{225}k_3+\frac{1}{180}k_4+\frac{1}{225}k_5-\frac{1}{2100}k_6\right)\right)$$

$$k_5=f\left(x_n+\frac{3}{4}h,y_n+h\left(\frac{17}{640}k_1+\frac{81}{350}k_2+\frac{7}{50}k_3+\frac{81}{320}k_4+\frac{41}{400}k_5-\frac{81}{22400}k_6\right)\right)$$

$$k_6=f\left(x_n+h,y_n+h\left(\frac{13}{270}k_1+\frac{256}{1575}k_2+\frac{256}{675}k_3-\frac{2}{45}k_4+\frac{256}{675}k_5+\frac{79}{1050}k_6\right)\right)$$

The method (17) is the second two-step-sixth stage implicit Runge -Kutta type method (TSIRK2) of this research.

Results and Discussions

Convergence analysis of TSIRK1 (14) and TSIRK2 (17) methods.

Theorem 1: *Super-convergence* ([9]).

If the condition $B(p)$ *holds for some* $p\geq s$, *then the collocation method has order* p *where*

$$B\left(p\right)=\sum_{i=1}^{s}b_ic_i^{k-1}=\tfrac{1}{k},k=1,2,\ldots,p \tag{18}$$

Applying (18) to (14) and (17), we see that both are of order

$p=6$. Hence, they are referred to as sixth-stage methods.
A Runge-Kutta method is consistent if and only if $\sum_{i=1}^{s} b_i = 1$ ([3]). Hence, both the pair (14) and (17) confirm the consistency of TSIRK1 and TSIRK2 respectively. Based on [3], the Runge-Kutta method is convergent if and only if it is consistent. Consequently, the methods TSIRK1 (14) and TSIRK2 (17) are both convergent [3].
Stability of the Runge-Kutta methods [3].

Definition 1: *The stability function* $R(z)$ *for* (13) *is defined by*

$$R(z)=\frac{\det(I-zA+zeb)}{\det(I-zA)}=\frac{N(z)}{D(z)} \qquad (19)$$

where $\{z\in \mathbb{C} : R(z)\le 1\}, e=(1,1,\dots,1)^T$.

Theorem 2: *A Runge-Kutta method with stability function* $R(z)=\frac{N(z)}{D(z)}$ *is A-stable if and only if (a) all poles of* R *(that is, all zeros of* D*) are in the right half-plane and (b)* $E(y)\ge 0$ for all real y where $E(y)=D(iy)D(-iy)-N(iy)N(-iy)$ (see [3] for the proof).
Applying (19) to (14) and (17) yield the stability polynomials

$$R_1(z)=\frac{\frac{1}{10368}z^5+\frac{97}{51480}z^4+\frac{347}{17280}z^3+\frac{73}{540}z^2+\frac{13}{24}z+1}{\frac{-1}{34560}z^5+\frac{1}{1296}z^4-\frac{7}{640}z^3+\frac{101}{1080}z^2-\frac{11}{24}z+1}$$ and

$$R_2(z)=\frac{\frac{7}{73728}z^5+\frac{341}{184320}z^4+\frac{613}{30720}z^3+\frac{259}{1920}z^2+\frac{13}{24}z+1}{-\frac{1}{40960}z^5+\frac{3}{4096}z^4-\frac{331}{30720}z^3+\frac{179}{1920}z^2-\frac{11}{24}z+1}$$ respectively.

Now, each of $R_1(z)$ and $R_2(Z)$ clearly satisfy theorem 2. In particular,

$$E_1(y)=\frac{91}{10749542400}y^{10}-\frac{15703717}{549543481344400}y^8-\frac{7081}{2001542400}y^6+\frac{97}{3706560}y^4\ge 0$$

and $E_2(y)=\frac{143}{16986931200}t^{10}-\frac{1}{2654208}t^8\ge 0$

respectively for all real y. Thus, TSRIK1 (14) and TSRIK2 (17) are both A-stable and are viable alternatives for solving stiff initial value problems as well.

Numerical experiments.
We test the accuracy of the new methods in terms of their absolute errors with some examples from the literature, and the results are presented in Tables 1–5, where we have used the notation $a(b):=a\times 10^b$.
Example 1 [11].

Solve $y'-2y=e^{-x}, y(0)=\frac{3}{4}, h=0.1, 0\le x\le 1$.

Exact solution: $y(x)=-\frac{1}{3}e^{-3x}+\frac{13}{12}e^{2x}$.

Table 1 compares the results in [11], and the smaller errors in Table 1 indicate that the new pair, TSIRK1 and TSIRK2, outperformed [11], with an outstanding result by TSIRK2.

Table 1: Absolute Errors for Example 1.

x	Mshelia *et al.* [11] $k=3$	TSIRK1 (14) $k=2$	TSIRK2 (17) $k=2$
01	1.4442E(-11)	6.8729E(-12)	5.0930E(-12)
0.2	3.5268E(-11)	1.6785E(-11)	1.2438E(-11)
0.3	6.4600E(-11)	3.0744E(-11)	2.2782E(-11)
0.4	1.0518E(-10)	5.0059E(-11)	3.7094E(-11)
0.5	1.6057E(-10)	7.6410E(-11)	5.6625E(-11)
0.6	2.0000E(-09)	1.1190E(-10)	8.2984E(-11)
0.7	3.0000E(-09)	1.5956E(-10)	1.1824E(-10)
0.8	1.0000E(-09)	2.2271E(-10)	1.6503E(-10)
0.9	1.0000E(-09)	3.0600E(-10)	2.2675E(-10)
1.0	6.0000E(-09)	4.1524E(-10)	3.0770E(-10)

Example 2 [2] Solve:

$y' = -8y + 8x + 1, y(0) = 2, h = 0.1, 0 \le x \le 1$.

Exact solution: $y(x) = x + 2e^{-8x}$. Table 2 compares the results with those in [1] and [2], and the smaller errors in Table 2 indicate better accuracy of the new methods over [1] and [2], with TSIRK2 being the best.

Table 2: Absolute Errors for example 2.

x	Agam [1]	Agam [2]	TSIRK1 (14)	TSIRK2 (17)
01	7.82E(-06)	1.89E(-06)	1.1497E(-07)	9.8582E(-08)
0.2	1.72E(-06)	1.69E(-06)	1.0332E(-07)	8.8591E(-08)
0.3	1.16E(-06)	1.14E(-06)	6.9638E(-08)	5.9710E(-08)
0.4	6.95E(-07)	6.80E(-07)	4.1721E(-08)	3.5772E(-08)
0.5	3.91E(-07)	3.85E(-07)	2.3433E(-08)	2.0092E(-08)

Example 3 [11], Solve:

$$y' = 20x^2 - 20y + 2x, y(0) = \frac{1}{3}, h = 0.05, 0 \le x \le 1.$$

Exact solution: $y(x) = x^2 + \frac{1}{3}e^{-20x}$. The results are compared with [8], [11], and [13] in Table 3, and the smaller errors in Table 3 also indicate better performance of the new pair, TSIRK1 and TSIRK2, over [8], [11], and [13], with TSIRK2 being the most accurate.

Table 3: Absolute Errors for Example 3.

x	Yakubu *et al.* [13] $k=3$	Kwami [8] $k=3$	Mshelia *et al.* [11] $k=3$	TSIRK1 (14) $k=2$	TSIRK2 (17) $k=2$
0.1	1.0629E(-02)	4.2107E(-05)	1.2554E(-07)	6.0252E(-08)	5.2655E(-08)
0.2	5.3890E(-03)	6.4073E(-04)	3.3979E(-08)	1.6317E(-08)	1.4252E(-08)
0.3	1.2320E(-02)	5.3864E(-04)	6.8979E(-09)	3.3123E(-09)	2.8932E(-09)
0.4	1.3008E(-03)	7.2861E(-05)	1.2447E(-09)	5.9770E(-10)	5.2207E(-10)
0.5	4.1148E(-04)	1.0418E(-05)	2.1056E(-10)	1.0111E(-10)	8.8319E(-11)
0.6	3.9430E(-04)	1.7999E(-06)	3.4196E(-11)	1.6421E(-11)	1.4343E(-11)
0.7	4.0724E(-05)	2.4356E(-07)	5.4000E(-12)	2.5928E(-12)	2.2647E(-12)
0.8	1.3629E(-05)	3.3443E(-08)	8.3600E(-13)	4.0112E(-13)	3.5039E(-13)
0.9	1.3677E(-05)	4.8625E(-09)	1.2800E(-13)	6.1062E(-14)	5.3291E(-14)
1.0	1.4145E(-06)	1.2680E(-07)	1.0000E(-14)	9.1038E(-15)	7.9936E(-15)

Example 4 [11] Solve:

$y' = -y, y(0) = 1, h = 0.05, 0 \le x \le 1$. Exact solution: $y(x) = e^{-x}$.

Table 4 compares the results in [8], [11], and [13] and the smaller errors in Table 4 again indicate the improved accuracy of the new pair with TSIRK2 outperforming.

Table 4: Absolute Errors for example 4

x	Yakubu *et al.* [13] $k=3$	Kwami [8] $k=3$	Mshelia *et al.* [11] $k=3$	TSIRK1 (14) $k=2$	TSIRK2 (17) $k=2$
01	2.1541E(-7)	1.0482E(-11)	2.0E(-15)	5.5511E(-16)	4.4409E(-16)
0.2	6.9544E(-7)	4.9204E(-11)	4.0E(-15)	1.1102E(-15)	7.7716E(-16)
0.3	2.8062E(-7)	5.6546E(-11)	6.0E(-15)	1.5543E(-15)	1.1102E(-15)
0.4	4.1350E(-7)	5.8933E(-11)	7.0E(-15)	1.8874E(-15)	1.3323E(-15)
0.5	2.8127E(-7)	8.2750E(-11)	7.0E(-15)	2.1094E(-15)	1.5543E(-15)
0.6	4.1578E(-7)	8.3785E(-11)	7.0E(-15)	2.4425E(-15)	1.7764E(-15)
0.7	4.9444E(-7)	8.1565E(-11)	9.0E(-15)	2.4425E(-15)	1.7764E(-15)
0.8	3.7785E(-7)	9.5601E(-11)	9.0E(-15)	2.4425E(-15)	1.7764E(-15)
0.9	4.6203E(-7)	9.3104E(-11)	9.0E(-15)	2.4980E(-15)	1.8319E(-15)
1.0	5.0564E(-7)	8.8506E(-11)	9.0E(-15)	2.5535E(-15)	1.8874E(-15)

Example 5 [10], Solve:

$y' = xy, y(0) = 1, h = 0.1, 0 \le x \le 1$. Exact solution:

$y(x) = e^{\frac{x^2}{2}}$.

The results are compared with [5] and [10] in Table 5. The smaller errors in Table 5 clearly demonstrate the improved accuracy of the new pair TSIRK1 and TSIRK2 methods developed in this research over [5] and [10] with TSIRK2 outperforming.

Table 5: Absolute Errors for Example 5.

x	James *et al.* [5].	Momoh *et al.* [10].	TSIRK1 (14)	TSIRK2 (17)
01	5.29E(-7)	2.6067E(-11)	2.3648E(-13)	1.9096E(-13)
0.2	1.77E(-7)	8.4319E(-11)	9.5901E(-13)	7.5540E(-13)
0.3	8.99E(-7)	1.8509E(-10)	2.2520E(-12)	1.7577E(-12)
0.4	3.09E(-7)	3.4796E(-10)	4.2693E(-12)	3.3151E(-12)
0.5	1.91E(-6)	6.0512E(-10)	7.2562E(-12)	5.6146E(-12)
0.6	4.48E(-6)	1.0070E(-09)	1.1582E(-11)	8.9369E(-12)
0.7	1.02E(-5)	1.6310E(-09)	1.7791E(-11)	1.3696E(-11)
0.8	7.74E(-5)	2.5956E(-09)	2.6682E(-11)	2.0499E(-12)
0.9	1.44E(-5)	4.0816E(-09)	3.9426E(-11)	3.0230E(-11)
1.0	2.93E(-5)	6.3647E(-09)	5.7738E(-11)	4.4190E(-11)

Conclusion

Results in Tables 1–5 show that the absolute errors of TSIRK1 (14) and TSIRK2 (17) of this research yielded the least errors when compared with the listed existing methods. This indicates better accuracy. Furthermore, the superiority of TSIRK2 over TSIRK1 is clearly seen. Thus, this pair could become viable alternatives for solving similar ODEs, as opposed to the existing methods in the literature. The A-stability property of the methods suggests their applicability to stiff ODES as well.

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References

[1] Agam, S. A. 2014. **A Sixth Order Multiply Implicit Runge-Kutta Method for the Solution of First and Second Order Ordinary Differential Equations**. Unpublished PhD Thesis, Department of Mathematics, Nigerian Defence Academy, Kaduna, Nigeria.

[2] Agam, S. A. and Yahaya, Y. A. 2014. **A highly efficient implicit Runge-Kutta method for first order ordinary differential equations.** *African Journal of Mathematics and Computer Science Research.* 7(5), 55-60. https://doi.org/10.5897/AJMCSR2014.0551 Accessed 8th November, 2018.

[3] Butcher, J. 2008. **Numerical Methods Ordinary Different Equations** (2nd ed). New York: John Wiley and Sons.

[4] Fang, Y., Hu, X. and Li, J. 2021. **Explicit pseudo two-step exponential Runge-Kutta methods for the numerical integration of first-order differential equations.** *Numerical Algorithms.* 86, 1143-1163. https://doi.org/10.1007/s11075-020-00927-4 Accessed 13th November, 2022.

[5] James, A. A., Adesanya, A.O., Sunday, J. and Yakubu, D. G. 2013. **Half-step continuous block method for the solution of modeled problems of ordinary differential equations**. *American Journal of Computational Mathematics.* 3, 261-269.

[6] Koroche, K. D. 2021. **Numerical Solution of First Order Ordinary Differential Equation by Using Runge-Kutta Method.** *International Journal of Systems Science and Applied Mathematics.* Vol. 6, No. 1, 2021, pp. 1-8. https://doi.org/10.11648/j.ijssam.20210601.11 Accessed 13th November, 2022

[7] Kendall, E. 1989. **An Introduction to Numerical Analysis,** (2nd ed). John Wiley and Sons. New York.

[8] Kwami. A. M. 2011. **A Special Class of Continuous General Linear Multistep Methods for Ordinary Differential Equations**. Unpublished PhD Thesis, Department of Mathematical Sciences, Abubakar Tafawa Balewa University, Bauchi, Nigeria.

[9] Landis, F. 2005. **Runge-Kutta and Collocation Methods**. Geometrical Numerical Integration http://www.sam.math.ethz.ch/~hiptmair/Seminars/GNI/slides/Landis.pdf. Accessed 16th July, 2019, pp 4-16.

[10] Momoh, A. A., Adesanya, A. O., Fasasi, K. M. and Tahir, A. 2014. **A new Numerical Integrator for the solution of stiff first order ordinary differential equations**. *Engineering Mathematics Letters*, Article ID 5. http://scik.org/index.php/eml/article/view/1589. Accessed 20th July, 2019.

[11] Mshelia, D. W., Yakubu, D. G., Badmus, A. M. and Manjak, N. H. 2016. **Derivation of three-step sixth stage Runge-Kutta method for the solution of first order differential equations**. *International Journal of Scientific and Technical Research in Engineering*, *1*(1)18-26. http://www.ijstre.com/Publish/042016/371428197.pdf. Accessed 10th July, 2018.

[12] Rattenbury, N. 2005. Almost Runge –Kutta Methods for Stiff and Non –stiff Problems. PhD Thesis (Unpublished). The University of Auckland, New Zealand. www.math.auckland.ac.nz/~butcher/theses/thesis-25-11-05.pdf. Accessed 10th June, 2018.

[13] Yakubu, D.G., Onumanyi, P. and Chollom, J. P. 2004. **A New Family of General Linear Methods based on the Block Adams-Moulton Multistep Methods**. *Science Forum*, *Journal of Pure and Applied Sciences,* 7(1), 98–106.

[14] Yakubu, D. G., Kwami, A. M. and Ahmed, M. L. 2012. A special class of continuous general linear methods. *Computational and Applied Mathematics*, 31(2), 259-281. https://www.scielo.br/j/cam/a/FVyX38RHDTDyPd9GSgFJKCS/?lang=en&format=pdf Accessed 10th July, 2018.

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