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One-Step Hybrid Block Method for the Numerical Solution of Malthus and SIR Epidemic Models equations

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Abstract

The Malthusian population model and Susceptible-Infectious-Recovered (SIR) epidemic model are described by systems of first order ordinary differential equations (ODEs) that model the dynamics of population growth and infectious disease spread respectively. Efficient numerical methods to solve these models play an integral role across mathematical biology disciplines. This study presents the formulation, analysis and application of a novel one-step hybrid block method for the direct integration of the Malthus and SIR models ODE systems.Consistency, zero-stability and convergence tests prove the method is numerically satisfactory. The technique is implemented in solving benchmark problems for the Malthusian quadratic and SIR epidemic models. Numerical results demonstrated that the new solver is more accurate and computationally efficient when compared to existing one-step and linear multistep methods widely used for population dynamics models. The hybrid block method provides an accurate and fast approach for simulating the dynamics of these biological systems.

Keywords: Malthus, SIR, Block Method, Implicit, Power Series.

Introduction

The Malthusian growth model and the SIR (susceptible-infected-recovered) epidemic model are two important differential equation models used in mathematical biologywith widespread applications.The Malthusian model describes population growth under limited resources, while the SIR model simulates the spread of infectious diseases in a population.Numerical methods to solve these models accurately and efficiently are of great interest.

In the spirit of that, we consider a numerical method for solving epidemic modeldifferential equation of first order initial value problems (IVPs) of ordinary differential equations (ODEs) of the form

$$y' = f(x,y), y(x_0) = y_0 \quad (1.1)$$

Numerical methods are required to solve these models when analytical solutions are notavailable. Recently, [13] proposed a one-step hybrid block method for numerically solving these models. Hybrid block methods are a type of linear multistepmethod that uses previous solution points to estimate the solution at the next point.Recently, [1] developed a new one-step hybrid block method for directly solving systems of first-order ordinary differential equations (ODEs) that describe Malthusian growth and SIR epidemic spread. One-step methods calculate the solution directly from previous points without intermediate steps. Hybrid block methods use different polynomial approximations over non-overlapping blocks and can be more efficient than linear multistep methods. The proposed scheme combines implicit and explicit methods to improve stability and efficiency. According to the numerical results by [13], this new technique shows "enhanced consistency, zero-stability, convergence, and high accuracy" for solving both the Malthus and SIR models. They demonstrate that their method is more accurate and computationally efficient compared to some existing techniques.

In this paper, we developed a new one-step hybrid block method to solve systems of first-order ordinary differential equations (ODEs) that model the Malthusian and SIR dynamics. One-step methods calculate the solution directly from the previous point without intermediate steps. Hybrid block methods use different polynomial approximations over non-overlapping blocks and have advantages over traditional linear multistep methods.The new numerical scheme will provide an efficient way to simulate important biological population models. Further testing on other systems of ODEs from physics and engineering will help expand the application of this numerical method.

Materials and Method

Derivation of One-Step Hybrid Block Method (HBM)

We consider the approximate solution in the form

$$y(x) = \sum_{j=0}^{m+t-1} a_j x^j \qquad (1.2)$$

with first derivative given as

$$y'(x) = \sum_{j=1}^{m+t-1} j a_j x^{j-1} = f(x, y) \qquad (1.3)$$

Where a_j are the parameters to be determined [3], in this method we both interpolate (1.2) and collocate (1.3) at the same point $x_{n+j}, j = 0\left(\frac{1}{4}\right)1$ and the continuous linear multistep method (CLMM) is in the form

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 \\
1 & x_{n+\frac{1}{4}} & x^2_{n+\frac{1}{4}} & x^3_{n+\frac{1}{4}} & x^4_{n+\frac{1}{4}} & x^5_{n+\frac{1}{4}} & x^6_{n+\frac{1}{4}} & x^7_{n+\frac{1}{4}} & x^8_{n+\frac{1}{4}} & x^9_{n+\frac{1}{4}} \\
1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} & x^7_{n+\frac{1}{2}} & x^8_{n+\frac{1}{2}} & x^9_{n+\frac{1}{2}} \\
1 & x_{n+\frac{3}{4}} & x^2_{n+\frac{3}{4}} & x^3_{n+\frac{3}{4}} & x^4_{n+\frac{3}{4}} & x^5_{n+\frac{3}{4}} & x^6_{n+\frac{3}{4}} & x^7_{n+\frac{3}{4}} & x^8_{n+\frac{3}{4}} & x^9_{n+\frac{3}{4}} \\
1 & x_{n+1} & x^2_{n+1} & x^3_{n+1} & x^4_{n+1} & x^5_{n+1} & x^6_{n+1} & x^7_{n+1} & x^8_{n+1} & x^9_{n+1} \\
0 & 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\
0 & 1 & x_{n+\frac{1}{4}} & x^2_{n+\frac{1}{4}} & x^3_{n+\frac{1}{4}} & x^4_{n+\frac{1}{4}} & x^5_{n+\frac{1}{4}} & x^6_{n+\frac{1}{4}} & x^7_{n+\frac{1}{4}} & x^8_{n+\frac{1}{4}} \\
0 & 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} & x^7_{n+\frac{1}{2}} & x^8_{n+\frac{1}{2}} \\
0 & 1 & x_{n+\frac{3}{4}} & x^2_{n+\frac{3}{4}} & x^3_{n+\frac{3}{4}} & x^4_{n+\frac{3}{4}} & x^5_{n+\frac{3}{4}} & x^6_{n+\frac{3}{4}} & x^7_{n+\frac{3}{4}} & x^8_{n+\frac{3}{4}} \\
0 & 1 & x_{n+1} & x^2_{n+1} & x^3_{n+1} & x^4_{n+1} & x^5_{n+1} & x^6_{n+1} & x^7_{n+1} & x^8_{n+1}
\end{bmatrix} \qquad (1.4)$$

Gaussian elimination technique is used in finding the values of $a's$ in (1.4) which are then substituted into (1.2) to produce a continuous implicit scheme of the form

$$y(t) = \sum_{j=0}^{1} \alpha_j(t) y_{n+j} + \beta_j(t) f_{n+j}, j = 0\left(\frac{1}{4}\right)1 \qquad (1.5)$$

$$\alpha_0 = \frac{51200}{27}t^9 - \frac{252928}{27}t^8 + \frac{528640}{27}t^7 - \frac{607360}{27}t^6 + \frac{416200}{27}t^5 - \frac{171724}{27}t^4 + \frac{40310}{27}t^3 - \frac{485}{3}t^2 + 1$$

$$\alpha_{\frac{1}{4}} = \frac{327680}{27}t^9 - \frac{1507328}{27}t^8 + \frac{2871296}{27}t^7 - \frac{2914304}{27}t^6 + \frac{1682432}{27}t^5 - \frac{541184}{27}t^4 + \frac{28672}{9}t^3 - \frac{512}{3}t^2$$

$$\alpha_{\frac{1}{2}} = 4096t^8 - 16384t^7 + 26112t^6 - 20992t^5 + 8848t^4 - 1824t^3 + 144t^2$$

$$\alpha_{\frac{3}{4}} = -\frac{327680}{27}t^9 + \frac{1441792}{27}t^8 - \frac{2609152}{27}t^7 + \frac{2504704}{27}t^6 - \frac{1371136}{27}t^5 + \frac{426496}{27}t^4 - \frac{69632}{27}t^3 + \frac{512}{3}t^2$$

$$\alpha_1 = -\frac{51200}{27}t^9 + \frac{207872}{27}t^8 - \frac{348416}{27}t^7 + \frac{311936}{27}t^6 - \frac{160712}{27}t^5 + \frac{47516}{27}t^4 - \frac{2482}{9}t^3 + \frac{53}{3}t^2$$

$$\beta_0 = \frac{1024}{9}t^9 - \frac{5120}{9}t^8 + \frac{10880}{9}t^7 - \frac{12800}{9}t^6 + \frac{9092}{9}t^5 - \frac{3980}{9}t^4 + \frac{1045}{9}t^3 - \frac{50}{3}t^2 + t$$

$$\beta_{\frac{1}{4}} = \frac{16384}{9}t^9 - \frac{77824}{9}t^8 + \frac{154624}{9}t^7 - \frac{166144}{9}t^6 + \frac{103936}{9}t^5 - \frac{37696}{9}t^4 + \frac{2432}{3}t^3 - 64t^2$$

$$\beta_{\frac{1}{2}} = 4096t^9 - 18432t^8 + 34304t^7 - 34048t^6 + 19344t^5 - 6248t^4 + 1056t^3 - 72t^2$$
$$\beta_{\frac{3}{4}} = \frac{16384}{9}t^9 - \frac{69632}{9}t^8 + \frac{121856}{9}t^7 - \frac{113408}{9}t^6 + \frac{60416}{9}t^5 - \frac{18368}{9}t^4 + \frac{2944}{9}t^3 - \frac{64}{3}t^2$$
$$\beta_1 = \frac{1024}{9}t^9 - \frac{4096}{9}t^8 + \frac{6784}{9}t^7 - \frac{6016}{9}t^6 + \frac{3076}{9}t^5 - \frac{904}{9}t^4 + \frac{47}{3}t^3 - t^2 \quad (1.6)$$

Substituting the above equations (1.6) i.e the values of the constant into (1.5), gives the continuous scheme in the form

$$y(t) = \begin{bmatrix} \frac{51200}{27}t^9 - \frac{252928}{27}t^8 + \frac{528640}{27}t^7 - \frac{607360}{27}t^6 \\ + \frac{416200}{27}t^5 - \frac{171724}{27}t^4 + \frac{40310}{27}t^3 - \frac{485}{3}t^2 + 1 \end{bmatrix} y_n + \begin{bmatrix} \frac{327680}{27}t^9 - \frac{1507328}{27}t^8 + \frac{2871296}{27}t^7 - \frac{2914304}{27}t^6 \\ + \frac{1682432}{27}t^5 - \frac{541184}{27}t^4 + \frac{28672}{9}t^3 - \frac{512}{3}t^2 \end{bmatrix} y_{n+\frac{1}{4}}$$
$$+ \begin{bmatrix} 4096t^8 - 16384t^7 + 26112t^6 \\ -20992t^5 + 8848t^4 - 1824t^3 + 144t^2 \end{bmatrix} y_{n+\frac{1}{2}} + \begin{bmatrix} -\frac{327680}{27}t^9 + \frac{1441792}{27}t^8 - \frac{2609152}{27}t^7 + \frac{2504704}{27}t^6 \\ -\frac{1371136}{27}t^5 + \frac{426496}{27}t^4 - \frac{69632}{27}t^3 + \frac{512}{3}t^2 \end{bmatrix} y_{n+\frac{3}{4}}$$
$$+ \begin{bmatrix} -\frac{51200}{27}t^9 + \frac{207872}{27}t^8 - \frac{348416}{27}t^7 + \frac{311936}{27}t^6 \\ -\frac{160712}{27}t^5 + \frac{47516}{27}t^4 - \frac{2482}{9}t^3 + \frac{53}{3}t^2 \end{bmatrix} y_{n+1} + \begin{bmatrix} \frac{1024}{9}t^9 - \frac{5120}{9}t^8 + \frac{10880}{9}t^7 - \frac{12800}{9}t^6 \\ + \frac{9092}{9}t^5 - \frac{3980}{9}t^4 + \frac{1045}{9}t^3 - \frac{50}{3}t^2 + t \end{bmatrix} f_n$$
$$+ \begin{bmatrix} \frac{16384}{9}t^9 - \frac{77824}{9}t^8 + \frac{154624}{9}t^7 - \frac{166144}{9}t^6 \\ + \frac{103936}{9}t^5 - \frac{37696}{9}t^4 + \frac{2432}{3}t^3 - 64t^2 \end{bmatrix} f_{n+\frac{1}{4}} + \begin{bmatrix} 4096t^9 - 18432t^8 + 34304t^7 - 34048t^6 \\ +19344t^5 - 6248t^4 + 1056t^3 - 72t^2 \end{bmatrix} f_{n+\frac{1}{2}}$$
$$+ \begin{bmatrix} \frac{16384}{9}t^9 - \frac{69632}{9}t^8 + \frac{121856}{9}t^7 - \frac{113408}{9}t^6 \\ + \frac{60416}{9}t^5 - \frac{18368}{9}t^4 + \frac{2944}{9}t^3 - \frac{64}{3}t^2 \end{bmatrix} f_{n+\frac{3}{4}} + \begin{bmatrix} \frac{1024}{9}t^9 - \frac{4096}{9}t^8 + \frac{6784}{9}t^7 - \frac{6016}{9}t^6 \\ + \frac{3076}{9}t^5 - \frac{904}{9}t^4 + \frac{47}{3}t^3 - t^2 \end{bmatrix} f_{n+1} \quad (1.7)$$

where $t = \frac{x - x_n}{h}$, evaluating the continuous hybrid scheme (1.7) at off-grid points $x_{n+j}, j = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ and its first derivative, after some algebraic manipulations and sorting we obtained one step hybrid block method (HBM) in the form of

$$
\begin{pmatrix}
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{pmatrix}
\begin{pmatrix}
y_{n+\frac{1}{8}}\\ y_{n+\frac{1}{4}}\\ y_{n+\frac{3}{8}}\\ y_{n+\frac{1}{2}}\\ y_{n+\frac{5}{8}}\\ y_{n+\frac{3}{4}}\\ y_{n+\frac{7}{8}}\\ y_{n+1}
\end{pmatrix}
=
\begin{pmatrix}
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1
\end{pmatrix}
\begin{pmatrix}
y_{n-\frac{1}{8}}\\ y_{n-\frac{1}{4}}\\ y_{n-\frac{3}{8}}\\ y_{n-\frac{1}{2}}\\ y_{n-\frac{5}{8}}\\ y_{n-\frac{3}{4}}\\ y_{n-\frac{7}{8}}\\ y_{n}
\end{pmatrix}
$$

$$
+h
\begin{pmatrix}
0&0&0&0&0&0&0&\frac{1070017}{29030400}\\
0&0&0&0&0&0&0&\frac{32377}{907200}\\
0&0&0&0&0&0&0&\frac{12881}{358400}\\
0&0&0&0&0&0&0&\frac{4063}{113400}\\
0&0&0&0&0&0&0&\frac{41705}{1161216}\\
0&0&0&0&0&0&0&\frac{401}{11200}\\
0&0&0&0&0&0&0&\frac{149527}{4147200}\\
0&0&0&0&0&0&0&\frac{989}{28350}
\end{pmatrix}
\begin{pmatrix}
f_{n-\frac{1}{8}}\\ f_{n-\frac{1}{4}}\\ f_{n-\frac{3}{8}}\\ f_{n-\frac{1}{2}}\\ f_{n-\frac{5}{8}}\\ f_{n-\frac{3}{4}}\\ f_{n-\frac{7}{8}}\\ f_{n}
\end{pmatrix}
+h
\begin{pmatrix}
\frac{2233547}{14515200}&-\frac{2302297}{14515200}&\frac{2797679}{14515200}&-\frac{31457}{181440}&\frac{1573169}{14515200}&-\frac{645607}{14515200}&\frac{156437}{14515200}&-\frac{33953}{29030400}\\
\frac{22823}{113400}&-\frac{21247}{453600}&\frac{15011}{113400}&-\frac{2903}{22680}&\frac{9341}{113400}&-\frac{15577}{453600}&\frac{953}{113400}&-\frac{119}{129600}\\
\frac{35451}{179200}&\frac{1719}{179200}&\frac{39967}{179200}&-\frac{351}{2240}&\frac{17217}{179200}&-\frac{7031}{179200}&\frac{243}{25600}&-\frac{369}{358400}\\
\frac{2822}{14175}&\frac{61}{28350}&\frac{4094}{14175}&-\frac{227}{2835}&\frac{1154}{14175}&-\frac{989}{28350}&\frac{122}{14175}&-\frac{107}{113400}\\
\frac{115075}{580608}&\frac{3775}{580608}&\frac{159175}{580608}&-\frac{125}{36288}&\frac{85465}{580608}&-\frac{24575}{580608}&\frac{5725}{580608}&-\frac{175}{165888}\\
\frac{279}{1400}&\frac{9}{5600}&\frac{403}{1400}&-\frac{9}{280}&\frac{333}{1400}&\frac{79}{5600}&\frac{9}{1400}&-\frac{9}{11200}\\
\frac{408317}{2073600}&\frac{24353}{2073600}&\frac{542969}{2073600}&\frac{343}{25920}&\frac{368039}{2073600}&\frac{261023}{2073600}&\frac{111587}{2073600}&-\frac{8183}{4147200}\\
\frac{2944}{14175}&-\frac{464}{14175}&\frac{5248}{14175}&-\frac{454}{2835}&\frac{5248}{14175}&-\frac{464}{14175}&\frac{2944}{14175}&\frac{989}{28350}
\end{pmatrix}
\begin{pmatrix}
f_{n+\frac{1}{8}}\\ f_{n+\frac{1}{4}}\\ f_{n+\frac{3}{8}}\\ f_{n+\frac{1}{2}}\\ f_{n+\frac{5}{8}}\\ f_{n+\frac{3}{4}}\\ f_{n+\frac{7}{8}}\\ f_{n+1}
\end{pmatrix}
\quad (1.8)
$$

Writing out (1.8) the block explicitly gives the following normalized discrete scheme in the form of

$$
y_{n+\frac{1}{8}} = y_n + \left[\frac{1070017}{29030400} f_n + \frac{2233547}{14515200} f_{n+\frac{1}{8}} - \frac{2302297}{14515200} f_{n+\frac{1}{4}} + \frac{2797679}{14515200} f_{n+\frac{3}{8}} - \frac{31457}{181440} f_{n+\frac{1}{2}} + \frac{1573169}{14515200} f_{n+\frac{5}{8}} - \frac{645607}{14515200} f_{n+\frac{3}{4}} + \frac{156437}{14515200} f_{n+\frac{7}{8}} - \frac{33953}{29030400} f_{n+1}\right]
$$

$$
y_{n+\frac{1}{4}} = y_n + h\left[\frac{32377}{907200} f_n + \frac{22823}{113400} f_{n+\frac{1}{8}} - \frac{21247}{453600} f_{n+\frac{1}{4}} + \frac{15011}{113400} f_{n+\frac{3}{8}} - \frac{2903}{22680} f_{n+\frac{1}{2}} + \frac{9341}{113400} f_{n+\frac{5}{8}} - \frac{15577}{453600} f_{n+\frac{3}{4}} + \frac{953}{113400} f_{n+\frac{7}{8}} - \frac{119}{129600} f_{n+1}\right]
$$

$$
y_{n+\frac{3}{8}} = y_n + h\left[\frac{12881}{358400} f_n + \frac{35451}{179200} f_{n+\frac{1}{8}} + \frac{1719}{179200} f_{n+\frac{1}{4}} + \frac{39967}{179200} f_{n+\frac{3}{8}} - \frac{351}{2240} f_{n+\frac{1}{2}} + \frac{17217}{179200} f_{n+\frac{5}{8}} - \frac{7031}{179200} f_{n+\frac{3}{4}} + \frac{243}{25600} f_{n+\frac{7}{8}} - \frac{369}{358400} f_{n+1}\right]
$$

$$
y_{n+\frac{1}{2}} = y_n + h\left[\frac{4063}{113400} f_n + \frac{2822}{14175} f_{n+\frac{1}{8}} + \frac{61}{28350} f_{n+\frac{1}{4}} + \frac{4094}{14175} f_{n+\frac{3}{8}} - \frac{227}{2835} f_{n+\frac{1}{2}} + \frac{1154}{14175} f_{n+\frac{5}{8}} - \frac{989}{28350} f_{n+\frac{3}{4}} + \frac{122}{14175} f_{n+\frac{7}{8}} - \frac{107}{113400} f_{n+1}\right]
$$

$$
y_{n+\frac{5}{8}} = y_n + h\left[\frac{41705}{1161216} f_n + \frac{115075}{580608} f_{n+\frac{1}{8}} + \frac{3775}{580608} f_{n+\frac{1}{4}} + \frac{159175}{580608} f_{n+\frac{3}{8}} - \frac{125}{36288} f_{n+\frac{1}{2}} + \frac{85465}{580608} f_{n+\frac{5}{8}} - \frac{24575}{580608} f_{n+\frac{3}{4}} + \frac{5725}{580608} f_{n+\frac{7}{8}} - \frac{175}{165888} f_{n+1}\right]
$$

$$y_{n+\frac{3}{4}} = y_n + h\begin{bmatrix} \frac{401}{11200} f_n + \frac{279}{1400} f_{n+\frac{1}{8}} + \frac{9}{5600} f_{n+\frac{1}{4}} + \frac{403}{1400} f_{n+\frac{3}{8}} \\ -\frac{9}{280} f_{n+\frac{1}{2}} + \frac{333}{1400} f_{n+\frac{5}{8}} + \frac{79}{5600} f_{n+\frac{3}{4}} + \frac{9}{1400} f_{n+\frac{7}{8}} - \frac{9}{11200} f_{n+1} \end{bmatrix}$$

$$y_{n+\frac{7}{8}} = y_n + h\begin{bmatrix} \frac{149527}{4147200} f_n + \frac{408317}{2073600} f_{n+\frac{1}{8}} + \frac{24353}{2073600} f_{n+\frac{1}{4}} + \frac{542969}{2073600} f_{n+\frac{3}{8}} \\ +\frac{343}{25920} f_{n+\frac{1}{2}} + \frac{368039}{2073600} f_{n+\frac{5}{8}} + \frac{261023}{2073600} f_{n+\frac{3}{4}} + \frac{111587}{2073600} f_{n+\frac{7}{8}} - \frac{8183}{4147200} f_{n+1} \end{bmatrix}$$

$$y_{n+1} = y_n + h\begin{bmatrix} \frac{989}{28350} f_n + \frac{2944}{14175} f_{n+\frac{1}{8}} - \frac{464}{14175} f_{n+\frac{1}{4}} + \frac{5248}{14175} f_{n+\frac{3}{8}} \\ -\frac{454}{2835} f_{n+\frac{1}{2}} + \frac{5248}{14175} f_{n+\frac{5}{8}} - \frac{464}{14175} f_{n+\frac{3}{4}} + \frac{2944}{14175} f_{n+\frac{7}{8}} + \frac{989}{28350} f_{n+1} \end{bmatrix} \tag{1.9}$$

and the normalized discrete scheme in the form of

$$A_1 Y_s = A_0 Y_{s-1} + h[B_1 F_s + B_0 F_{s-1}] \tag{2.0}$$

where s represents the block number,

$$Y_s = \left(y_{n+\frac{1}{8}}, y_{n+\frac{1}{4}}, y_{n+\frac{3}{8}}, y_{n+\frac{1}{2}}, y_{n+\frac{5}{8}}, y_{n+\frac{3}{4}}, y_{n+\frac{7}{8}}, y_{n+1}\right)^T, Y_{s-1} = \left(y_{n-\frac{1}{8}}, y_{n-\frac{1}{4}}, y_{n-\frac{3}{8}}, y_{n-\frac{1}{2}}, y_{n-\frac{5}{8}}, y_{n-\frac{3}{4}}, y_{n-\frac{7}{8}}, y_n\right)^T,$$

$$F_s = \left(f_{n+\frac{1}{8}}, f_{n+\frac{1}{4}}, f_{n+\frac{3}{8}}, f_{n+\frac{1}{2}}, f_{n+\frac{5}{8}}, f_{n+\frac{3}{4}}, f_{n+\frac{7}{8}}, f_{n+1}\right)^T, F_{s-1} = \left(f_{n-\frac{1}{8}}, f_{n-\frac{1}{4}}, f_{n-\frac{3}{8}}, f_{n-\frac{1}{2}}, f_{n-\frac{5}{8}}, f_{n-\frac{3}{4}}, f_{n-\frac{7}{8}}, f_n\right)^T,$$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \frac{2233547}{14515200} & -\frac{2302297}{14515200} & \frac{2797679}{14515200} & -\frac{31457}{181440} & \frac{1573169}{14515200} & -\frac{645607}{14515200} & \frac{156437}{14515200} & -\frac{33953}{29030400} \\ \frac{22823}{113400} & -\frac{21247}{453600} & \frac{15011}{113400} & -\frac{2903}{22680} & \frac{9341}{113400} & -\frac{15577}{453600} & \frac{953}{113400} & -\frac{119}{129600} \\ \frac{35451}{179200} & \frac{1719}{179200} & \frac{39967}{179200} & -\frac{351}{2240} & \frac{17217}{179200} & -\frac{7031}{179200} & \frac{243}{25600} & -\frac{369}{358400} \\ \frac{2822}{14175} & \frac{61}{28350} & \frac{4094}{14175} & -\frac{227}{2835} & \frac{1154}{14175} & -\frac{989}{28350} & \frac{122}{14175} & -\frac{107}{113400} \\ \frac{115075}{580608} & \frac{3775}{580608} & \frac{159175}{580608} & -\frac{125}{36288} & \frac{85465}{580608} & -\frac{24575}{580608} & \frac{5725}{580608} & -\frac{175}{165888} \\ \frac{279}{1400} & \frac{9}{5600} & \frac{403}{1400} & -\frac{9}{280} & \frac{333}{1400} & \frac{79}{5600} & \frac{9}{1400} & -\frac{9}{11200} \\ \frac{408317}{2073600} & \frac{24353}{2073600} & \frac{542969}{2073600} & \frac{343}{25920} & \frac{368039}{2073600} & \frac{261023}{2073600} & \frac{111587}{2073600} & -\frac{8183}{4147200} \\ \frac{2944}{14175} & -\frac{464}{14175} & \frac{5248}{14175} & -\frac{454}{2835} & \frac{5248}{14175} & -\frac{464}{14175} & \frac{2944}{14175} & \frac{989}{28350} \end{pmatrix}, B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1070017}{29030400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32377}{907200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{12881}{358400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4063}{113400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{41705}{1161216} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{401}{11200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{149527}{4147200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{989}{28350} \end{pmatrix}$$

Order and Error Constant of the Block

Definition: A block linear multistep method of first order ODEs is said to be of order P if $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \cdots = \bar{c}_p = 0, \bar{c}_{p+1} \neq 0$. Thus, $\left[y_{n+\frac{1}{8}}, y_{n+\frac{1}{4}}, y_{n+\frac{3}{8}}, y_{n+\frac{1}{2}}, y_{n+\frac{5}{8}}, y_{n+\frac{3}{4}}, y_{n+\frac{7}{8}}, y_{n+1}\right]^T$ are of order $[9,9,9,9,9,9,9,10]^T$ and $\begin{bmatrix} \frac{8183}{1113255523123200}, \frac{9}{1503238553600}, \frac{25}{3848290697216}, \frac{47}{7610145177600}, \\ \frac{25}{3848290697216}, \frac{9}{1503238553600}, \frac{8183}{1113255523123200}, -\frac{37}{62783697715200} \end{bmatrix}^T$ respectively

c's are the coefficients of h and y functions while c_{p+1} is the error constant [10] and [14]. For our one-step method, expanding the block in Taylor series expansion we obtained

Zero Stability of the Block Method

Given the general form of block method $A^{(0)}Y_m = A^{(i)}Y_{m-1} + h^{\mu}\left[B^{(i)}F_m + B^{(0)}F_{m-1}\right]$

A block method is said to be zero stable, if the roots $det\left[\lambda A^{(0)} - A^{(i)}\right] = 0$ of the first characteristic polynomial satisfy $|\lambda| \leq 1$ and for the roots with $|\lambda| \leq 1$, the multiplicity must not exceed the order of the differential equation [12]. For our method,

$$\lambda \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} - \begin{pmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} =$$

$$\begin{pmatrix} \lambda&0&0&0&0&0&0&-1 \\ 0&\lambda&0&0&0&0&0&-1 \\ 0&0&\lambda&0&0&0&0&-1 \\ 0&0&0&\lambda&0&0&0&-1 \\ 0&0&0&0&\lambda&0&0&-1 \\ 0&0&0&0&0&\lambda&0&-1 \\ 0&0&0&0&0&0&\lambda&-1 \\ 0&0&0&0&0&0&0&\lambda-1 \end{pmatrix}$$

, determinant: $\lambda^8 - \lambda^7 = \lambda^7(\lambda - 1) = 0$, Hence the block is zero stable

Region of Absolute Stability

From equation (2.0), to obtain the region of absolute stability we apply the formula in the form of

$$[A_1]w - [A_0] - z[B_0]w - z[B_1]w \qquad (2.1)$$

after substituting the values of $A_1, A_0, B_1,$ and B_0, simplifying, taking the determinant and lastly sorting for z, we obtained

$$[A_1]w - [A_0] - z[B_0]w - z[B_1]w = -z^7\left(\frac{281}{660602880}w^8 + \frac{199}{1321205760}w^7\right) - z^6\left(\frac{1566947}{416179814400}w^7 - \frac{1566947}{416179814400}w^8\right) - z^5\left(\frac{590221}{3251404800}w^8 + \frac{585763}{6502809600}w^7\right) - z^4\left(\frac{3531473}{2786918400}w^7 - \frac{3531473}{2786918400}w^8\right) - z^3\left(\frac{38407}{1814400}w^8 + \frac{50761}{3628800}w^7\right) - z^2\left(\frac{366679}{3628800}w^7 - \frac{366679}{3628800}w^8\right) - z\left(\frac{7582}{14175}w^8 + \frac{6593}{14175}w^7\right) + w^8 - w^7$$

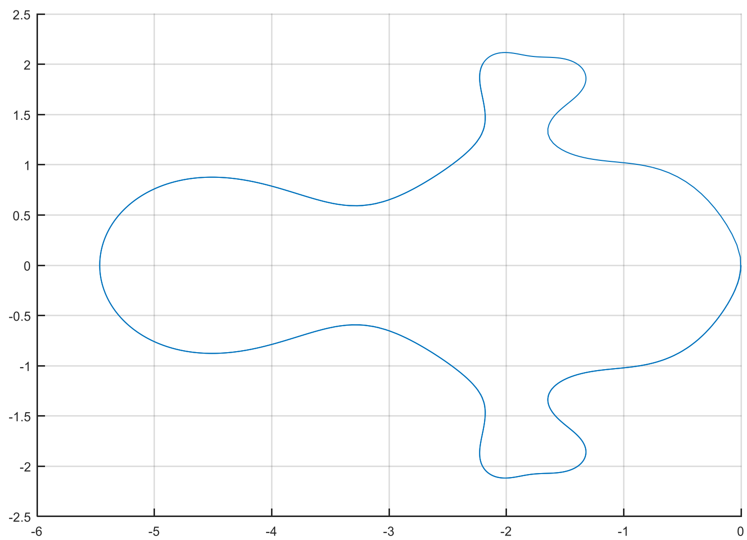

Fig 1: Region of Absolute Stability

Consistency and Convergence

Our new block method is consistent since the order of each of the method is greater than I [8]

Theorem 1: Convergence

According to [6], the necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. Also, [5], substantiated Lambert's stand on convergence as thus;
convergence = consistency + zero-stability. Hence the new block method is convergent.

Numerical examples

This section details a comparison between the solution of the one-step hybrid block method with [2] and [9]. The one-step hybrid block method was chosen for comparison as it has the least step number among the developed methods. Here, the performance of the new block method is examined on Malthus and SIRmodel. The results obtained from the test examples are shown in tabular form. We used MATLAB codes for the computational purposes.

Test Example 1: (SIR model)

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled equations relating the number of susceptible people S(t), number of people infected I(t), and the number of people who have recovered R(t). This is a good and simple model for many infectious diseases including measles, mumps and rubella. It is given by the following three coupled equations:

$$\begin{Bmatrix} \frac{dS}{dt} = \mu(1-S) - \beta IS \\ \frac{dI}{dt} = -\mu I - \gamma I + \beta IS \\ \frac{dR}{dt} = -\mu R + \gamma I \end{Bmatrix} \qquad (2.2)$$

Where μ, γ, β are positive parameters to be determined, define y to be, $y = S + I + R$ and adding the equations in (16) above, we obtain the following evolution equation for y, $y' = \mu(1-y)$. taking $\mu = 0.5$ and attaching an initial condition $y(0) = 0.5$ (for a particular closed population). We obtain,

$$\frac{dy}{dt} = 0.5(1-y), y(0) = 0.5$$

whose exact solution is

$$y(t) = 1 - 0.5e^{-0.5t}$$

source: [15]

Table 1: Comparison of result, of example 1

X	Exact Solution	Computed Solution	New Error (HBM)	[2]
0.1	0.52438528774964e	0.52438528774964e	0.0000000000000	9.103829e-015
0.2	0.54758129098202e	0.54758129098202e	0.0000000000000	7.105427e-015
0.3	0.56964601178747e	0.56964601178747e	0.0000000000000	8.881784e-015
0.4	0.59063462346101e	0.59063462346101e	1.1102230246e-16	2.120526e-014
0.5	0.61059960846430e	0.61059960846430e	0.0000000000000	1.367795e-013
0.6	0.62959088965914e	0.62959088965914e	1.1102230246e-16	7.982504e-013
0.7	0.64765595514064e	0.64765595514064e	1.1102230246e-16	3.698819e-012

Test Problem 2 (GROWTH model)

A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture; and after four hours, 3000 strands. Find the number of strands of the bacteria present in the culture at time $t\ :\ \ 0 \le t \le 1$. Let $N(t)$, denote the number of bacteria strands in the culture at time t, the initial value problem modeling this problem is given by,

$$\frac{dN}{dt} = 0.366N, N(0) = 694$$

the exact solution is given by

$$N(t) = 694e^{0.366t}$$

source: [15]

Table 2: Comparison of result, of example 2

X	Exact Solution	Computed Solution	New Error (HBM)	[2]
0.1	719.87095048413e	719.8709504841319e	1.1368683772e-13	6.821210e-013
0.2	746.70631894946e	746.7063189494633e	0.0000000000000	6.821210e-013
0.3	774.54205699518e	774.5420569951837e	0.0000000000000	6.821210e-013
0.4	803.41545642516e	803.4154564251550e	1.1368683772e-13	3.410605e-013
0.5	833.36519920816e	833.3651992080966e	0.0000000000000	7.503331e-012
0.6	864.43140930019e	864.4314093001880e	1.1368683772e-13	5.024958e-011
0.7	896.65570639952e	896.6557063995158e	1.1368683772e-13	2.373781e-010

Test Problem 3 (Malthus Growth Model)

Consider the exponential population growth rate problem

$$\frac{dp}{dt} = kp, t \in [0,1],$$

with the exact solution given by

$p(t) = 100e^{0.250679566129t}$, initial condition $p(0) = 100$ with $k = 0.250679566129$ and $h = 0.1$. Comparison with the two-step block method given in Table 3 below.
Source: [9]

Table 3: Comparison of result, of example 3

X	Exact Solution	Absolute Error (HBM)	[9]
0.1	102.5384799834733e	0.000000000000000	1.667719e-08
0.2	105.1413987732115e	0.000000000000000	4.400247e-10
0.3	107.8103921354134e	0.000000000000000	1.711722e-08
0.4	110.5471373598749e	0.000000000000000	8.800495e-10
0.5	113.3533543140580e	1.4210854715202e-14	1.755724e-08
0.6	116.2308065239160e	1.4210854715202e-14	1.320074e-09
0.7	119.1813022821552e	1.4210854715202e-14	1.799727e-08
0.8	122.2066957846305e	0.0000000000000000	1.760099e-09
0.9	125.3088882955874e	1.4210854715202e-14	1.843729e-08
1.0	128.4898293424838e	0.0000000000000000	2.200124e-09

Discussion of Result

From the numerical solution of the three test problems (SIR, Growth and Malthus Growth model) solved by the new one-step hybrid method, the result displayed in table (1) showed the superiority of the new block method over that of [2]. In table (2), it is clearly observed that the new one-step hybrid block method outperforms the method of [2] in terms of accuracy for the solution of example 2. Likewise, in example 3, the result shown that our new one-step hybrid methodin table 3, performed better than [9] in terms of consistent and accuracy.

Conclusion

The authors derived a new one-step hybrid block method and prove it is consistent, zero-stable, and convergent. They implement the new method to solve test problems for the Malthus and SIR models and compare the accuracy and efficiency to existing numerical methods. The results indicated that, the new method provides an accurate solution at low computational cost that is more efficient than other one-step and linear multistep methods.

References

[1] Ababneh, J. M., Ozioko, A.U., and Anake, T. A. (2022). **A new one-step hybrid block method for direct solution of the Malthusian and SIR epidemic models**. *Applied Mathematics and Computation*, 424, 127002.https://doi.org/10.1016/j.amc.2022.127002

[2] Abolarin, O. E., Ogunware, G. B. and Akinola, L. S. (2020). **An Efficient Seven-Step Block Method for Numerical Solution of SIR and Growth Model**. *FUOYE Journal of Engineering and Technology (FUOYEJET), Volume 5, Issue 1, March 2020 ISSN: 2579-0625 (Online), 2579-0617 (Paper)*

[3] Adee, S.O, and Yunusa S. (2022). **Some new hybrid block methods for solving non-stiff Initial value problems of ordinary differential equations**. *Nigerian Annals of Pure & Appl Sci. 5(1):265-279. DOI: 10.5281/zenodo.7135518*

[4] Adesanya, A. O., Odekunle, M. R., Alkali, M. A., and Abubakar, A. B. (2013). **Starting the five steps Stomer-Cowell method by Adams-Bashforth method for the solution of the first order ordinary differential equations**. *African Journal of Mathematics and Computer Science Research, Vol. 6(**5**), pp. 89-93*

[5] Akinfenwa, O. A., Ogunseye, H. A. and Okunuga, S. A. (2016). **Block Hybrid Method for Solution of Fourth Order Ordinary Differential equations**, *Nigerian Journal of Mathematics and Applications*, 25, 140 – 150.

[6] Lambert, J. D. (1973). **Computational methods in ordinary differential equations**. John Wiley and Sons, New York.

[7] Ikhile, M. N. (2015). **Numerical solution of SIR epidemiological model using fourth order Runge-Kutta method.** *Applied Mathematics*, 6(1), 67-72.https://doi.org/10.5923/j.am.20150601.09

[8] Olanegan, O. O., Ogunware, B. G., Omole E. O. Oyinloye, T. S. and Enoch, B. T. (2015). **Some Variable Hybrids Linear Multistep Methods for Solving First Order Ordinary Differential Equations Using Taylor's Series;** *IOSR Journal of Mathematics* 11, 08-13.

[9] Oluwaseun, A. and Zurni O. (2022). **Investigating accuracy of multistep block methods based on step-length values for solving Malthus and Verhulst growth models**. *AIP Conference Proceedings 2472, 030003; https://doi.org/10.1063/5.0092725Published Online: 19 August 2022*

[10] Omar, Z. and Abeldrahim, R. (2016). **New Uniform Order Single Step Hybrid Block Method for Solving Second Order Ordinary Differential Equations**. *International Journal of Applied Engineering Research 11(4), 2402-2406.*

[11] Omar, Z. and Adeyeye, O. (2016). **Numerical Solution of First Order Initial Value Problems Using a Self-Starting Implicit Two-Step Obrechkoff-Type Block Method**. *Journal of Mathematics and Statistics 12(2):127-134.*

[12] Omar, Z. and Kuboye, J. O. (2015). **Computation of an Accurate Implicit Block Method for Solving Third Order Ordinary Differential Equations Directly**. *Global Journal of Pure and Applied Mathematics.11,177-186.*

[13] Omeje, M., Ibrahim, A., and Sirisena, U. (2021). **One-step hybrid block method for the solution of malthus and SIR model equations**. *Mathematical Biosciences and Engineering, 18(6), 7687–7710.https://doi.org/10.3934/mbe.2021417*

[14] Omole, E. O. and Ogunware, B. G. (2018). **3-Point Single Hybrid Block Method (3PSHBM) for Direct Solution of General Second Order Initial Value Problem of Ordinary Differential Equations**. *Journal of Scientific Research & Reports 20(3): 1-11; ISSN: 2320-0227.*

[15] Sunday, J., Odekunle M.R. and Adesanya, A.O. (2013). **Order six block integrator for the solution of first order ordinary differential equations**. *Int. J. Math. Soft Comput., 3: 87-96*

.

Yunusa S., Adee S.O., Kida M & Etuk E.D. (2024). One-Step Hybrid Block Method for the Numerical Solution of Malthus and SIR Epidemic Models equations. FUAM Journal of Pure and Applied Science, **4(1)**:95-103