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Projection Methods for the solution of Volterra Integro-Differential Equations

*1M.M. Shior, 3T. Aboiyar, 2S.O. Adee and 3E.C. Madubueze
1Department of Mathematics/Computer science Benue State University, Makurdi.
2Department of Mathematics Modibo Adama University of Technology, Yola
3 Department of Mathematics Joseph Sarwuan Tarka University, Makurdi
***Correspondence E-mail:** mshior@bsum.edu.ng

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Abstract

In this paper, we utilized the linear generalized inverse multiquadric function and the quadratic generalized multiquadric function as radial basis functions for the quadrature-based projection method in solving Volterra integro-differential equations (VIDE) of first and second orders. The selected examples are evaluated using MAPLE 17 and MATLAB Softwares and the obtained results compared with the exact solution for the linear generalized inverse multiquadric function and the quadratic generalized linear multiquadric function as radial basis functions. The quadratic generalized linear multiquadric function has the best approximation for both the first and second order VIDEs.

Keywords: Projection, Quadrature, Integro-Differential, Linear Multiquadric, Quadratic multiquadric and Radial basis

Introduction

Projection approximation methods play an essential role in approximation theory, and have many interesting applications, particularly in solving integral equations. D' Almeida and Fernandes [3] used the projection approximation for solving weakly singular Fredholm integral equations of the second kind. For the solution of a weakly singular Fredholm integral equation of the second kind defined on a Banach space, for instance $L^1(a, b)$, the classical projection methods with the discretization of the approximating operator on a finite dimensional subspace usually use a basis of this subspace built with grids on (a, b) . This may require a large dimension of the subspace. One way to overcome this problem is to include more information in the approximating operator or to compose one classical method with one step of iterative refinement. This is the case of Kulkarni method or iterated Kantorovich method. Here they compared these methods in terms of accuracy and arithmetic workload. A theorem stating comparable error bounds for these methods, under very weak assumptions on the kernel, the solution and the space where the problem is set, was given. According to [10], [6], [5], [1], and [9], in recent years collocation methods such as Cheby-shev, Taylor polynomials and B-spline functions have been given for approximating the solutions of linear Fredholm–Volterra integro-differential equations. There are various methods to get the approximate solution. The most popular of these is the collocation method. Moreover, according to [2], these methods can be called projection methods because the collocation method makes essential use of projection (linear) operators. Mennouni and Guedjiba [7] studied projection approximations for solving Cauchy integro-differential equations using airfoil polynomials of the first kind. They studied a more general case and proved the convergence of the method. The proposed method was tested for two kernels which are particularly important in practice, some numerical examples were used to illustrate the accuracy of the method. Mennouni and Guedjiba [8] used the projection method to investigate the numerical solution for a class of integro-differential equations with Cauchy kernel by using airfoil polynomials of the first kind as basis function. According to them, this method, we obtained a system of linear algebraic equations and give some sufficient conditions for the convergence of the method. They Finally, investigated the computational performance of the method through some numerical examples.

Based on the above, we will utilize the linear generalized inverse multiquadric function and the quadratic generalized linear multiquadric function as radial basis functions for the quadrature-based projection method in solving Volterra integro-differential equations of the first and second orders.

The Projection Method

We consider the nth order integro-differential equations of the form;

$$\sum_{n=0}^{N} P_n(x)u^{(n)}(x) = F(x) + \lambda \int_a^x K(x,t)u(y)dy \quad (1)$$

Where the known functions $P_n(x), f(x), K(x,t)$ are defined on $x, y \in [a,b]$; λ a real parameter, $u(x)$ is the unknown function.

We will assume we have k initial conditions;

$$u(x_0)=\alpha_1, u'(x_0)=\alpha_2, u^{(k-1)}(x_0)=\alpha_k \quad (2)$$

In projection methods, we consider solving (3.95) within a frame work of some complete function space V, we choose a sequence of finite dimensional approximating subspaces $V_n \subset V, n=1$ with V_n having dimensionsk_n. Let V_n have a basis $\{\phi_1,\cdots,\phi_N\}$, with $N=N_n$. We seek a functions $u_n \in V_n$ which can be written as

$$u_n(x)=\sum_{j=1}^{N} C_j\phi_j(x). \quad (3)$$

This is substituted into (1), and the coefficients $\{C_1,\cdots,C_N\}$ are determined by forcing the equation to be almost exact in some sense. We will introduce

$$\sigma_n(x)=\sum_{n=0}^{N} P_n(x)u^{(n)}(x)-F(x)-\lambda\int_a^x K(x,t)u(y)dy$$
$$=\sum_{j=1}^{N} C_j\left(\sum_{n=0}^{k} P_n(x)\phi_j^{(n)}(x)-\int_a^x K(x,t)\phi_j(y)dy\right) - F(x) \quad (4)$$

For $x\in[a,b]$. This equation is called the residual in the approximation of the equation when using $u\approx u_n$.
The coefficients $\{C_1,\cdots,C_N\}$ are chosen by forcing $\sigma_n(x)$ to be approximately zero in some sense. The hope and expectation is that the resulting function $u_n(x)$ will be a good approximation of the true solution.
We will pick distinct node points $x_1,\cdots,x_N\in[a,b]$. $\sigma_n(x_i)=0, i=1,2,\cdots,N$
This leads to determining $\{C_1,\cdots,C_N\}$ as the solution of the linear system.

$$\sum_{j=1}^{N} C_j\left(\sum_{n=0}^{k} P_n(x_i)\phi_j^{(n)}(x_i)-\int_a^x K(x_i,t)\phi_j(y)dy\right) - F(x_i) \quad (5)$$
$i=1,2,\cdots,N$.

For solving a k^{th} order equation, then for N unknown, and k initial conditions. We consider N data sites. We select $N-k$ point to evaluate (5) while the remaining equations are obtained using

$$u_N^{(m)}(x_0)=\alpha_{m+1}, \qquad m=0,\cdots,k-1 \quad (6)$$

The integral domain $[a,x]$ must be transferred to the fixed interval $[a,b]$. For this purpose, the following transformation can be used.

$$P(x,\theta)=\frac{x-a}{b-a}\theta+\frac{b-x}{b-a}a$$

Employing this transformation (5) becomes

$$\sum_{j=1}^{N} C_j\left(\sum_{n=0}^{k} P_n(x_i)\phi_j^{(n)}(x_i) - \int_a^b K^*\big(x_i,P(x_i,\theta)\big)\phi_j(\theta)d\theta\right) = F(x_i),$$
$$i=1,2,\cdots,N-k \quad (7)$$

Where $K^*=\frac{x-a}{b-a}K$
An m-point quadrature formula with coefficients τ_s and weight w_s in the interval $[a,b]$ in (7) yields

$$\sum_{j=1}^{N} C_j\left(\sum_{n=0}^{k} P_n(x_i)\phi_j^{(n)}(x_i) - \sum_{s=1}^{m} w_sK^*\big(x_i,P(x_i,\tau_s)\big)\phi_j(\tau_s)\right) = F(x_i),$$
$$i=1,2,\cdots,N-k \quad (8)$$

For our choice of the basis function, $\phi_j(x)$ form a data-dependent space. To this end, we use a function of the form $\phi_j(x)=\phi(x,x_j)=\phi(|x-x_j|)$.

In this work we will consider
$\phi(r)=\frac{2-(\varepsilon r)^2}{(1+(\varepsilon r)^2)^4}$ (linear generalized inverse multiquadric function)
$\phi(r)=\frac{3-6(\varepsilon r)^2+(\varepsilon r)^4}{(1+(\varepsilon r)^2)^6}$ (quadratic generalized multiquadric function)

Application of the quadrature based projection method using the linear generalized inverse multiquadric

Example 1 Consider the Volterra integro-differential equation of the first order in Day (1967) given by:

$$\left.\begin{aligned} y'(x)+y(x)=(x^2+2x+1)e^{-x} \\ +5x^2+8-\int_0^x ty(t)dt, \\ 0\le x\le 1 \\ y(0)=10 \end{aligned}\right\} \quad (9)$$

The exact solution is $y(x)=10-xe^{-x}$.

Let

$$u_n(x)=\sum_{j=1}^{N} C_j\phi_j(x) \quad (10)$$

and $\phi(r)=\frac{2-(\varepsilon r)^2}{(1+(\varepsilon r)^2)^4}$.

By equation (8) that is;

$$\sum_{j=1}^{N} C_j \left(\sum_{n=0}^{k} P_n(x_i)\phi_j{}^{(n)}(x_i) - \int_a^b K^*\big(x_i, P(x_i,\theta)\big)\phi_j(\theta)d\theta \right) = F(x_i)\,, \qquad i = 1,2,\cdots,N-k$$

and considering the integro-differential equation in Example 1, we have:
$K^* = \frac{x-a}{b-a}K,$
where $\phi_j(x)$ are basis functions. Comparing (8) with (9) we have
$K(x,t) = t\,,$ $F(x) = (x^2+2x+1)e^{-x} + 5x^2 + 8$, $P(x,\theta) = \frac{x-a}{b-a}\theta + \frac{b-x}{b-a}a\,.$
Using seventeen collocation points and simplifying with the aid MATLAB software we have the result of the method compared with the exact solution in Table 1.

Table 1: The Projection Method for Example 1 with the Linear Generalized Inverse Multiquadric and Quadratic Generalized Inverse Multiquadric with Absolute Errors

i	x_i	Solution with Linear Generalized Inverse Multiquadric	Exact solution	Absolute error	Solution with Quadratic Generalized Inverse Multiquadric	Absolute error
1	0.000000	9.999990	10.000000	7.781864×10^{-9}	10.000000	2.548788×10^{-9}
2	0.055556	9.947456	9.947447	9.682569×10^{-6}	9.947475	2.834905×10^{-5}
3	0.111111	9.900583	9.900573	9.379252×10^{-6}	9.900602	2.876289×10^{-5}
4	0.166667	9.858928	9.858920	8.782200×10^{-6}	9.858946	2.647638×10^{-5}
5	0.222222	9.822067	9.822058	8.337886×10^{-6}	9.822084	2.532593×10^{-5}
6	0.277778	9.789601	9.789593	7.859580×10^{-6}	9.789617	2.375425×10^{-5}
7	0.333333	9.761164	9.761156	7.414458×10^{-6}	9.761179	2.250079×10^{-5}
8	0.388889	9.736414	9.736407	6.981155×10^{-6}	9.736429	2.112758×10^{-5}
9	0.444444	9.715038	9.715031	6.577881×10^{-6}	9.715051	1.993287×10^{-5}
10	0.500000	9.696741	9.696735	6.165623×10^{-6}	9.696753	1.866418×10^{-5}
11	0.555556	9.681254	9.681248	5.792774×10^{-6}	9.681266	1.754303×10^{-5}
12	0.611111	9.668326	9.668321	5.394031×10^{-6}	9.668337	1.633596×10^{-5}
13	0.666667	9.657727	9.657722	5.054930×10^{-6}	9.657737	1.530887×10^{-5}
14	0.722222	9.649242	9.649237	4.677106×10^{-6}	9.649251	1.411580×10^{-5}
15	0.777778	9.642673	9.642668	4.350966×10^{-6}	9.642682	1.324219×10^{-5}
16	0.833333	9.637839	9.637835	3.959464×10^{-6}	9.637847	1.184735×10^{-5}
17	0.888889	9.634571	9.634567	3.758833×10^{-6}	9.634579	1.178639×10^{-5}
18	0.944444	9.632713	9.632710	2.884820×10^{-6}	9.632717	7.462701×10^{-6}
19	1.000000	9.632127	9.632121	6.364444×10^{-6}	9.632147	2.675520×10^{-5}

Example 2

Let us consider the Volterra integro-differential equation of the second order in Wazwaz (2011) given by

$$\left. \begin{array}{c} y''(x) = (1+x) + \int_0^x (x-t)y(t)dt\,, \\ 0 \le x \le 1 \\ y(0) = 1 \\ y'(0) = 1 \end{array} \right\} \qquad (11)$$

The exact solution is $y(x) = e^x$.

let

$$u_n(x) = \sum_{j=1}^{N} C_j\phi_j(x) \qquad (12)$$

By equation (8) that is

$$\sum_{j=1}^{N} C_j \left(\sum_{n=0}^{k} P_n(x_i)\phi_j{}^{(n)}(x_i) - \int_a^b K^*\big(x_i, P(x_i,\theta)\big)\phi_j(\theta)d\theta \right) = F(x_i)\,, \qquad i = 1,2,\cdots,N-k$$

and considering the integro-differential equation in Example 2, we have:

$K^* = \frac{x-a}{b-a}K,$
where $\phi_j(x)$ are basis functions. comparing (8) with (11) we have

$K(x,t) = x - t$, $F(x) = (1 + x)$, $P(x,\theta) = \frac{x-a}{b-a}\theta + \frac{b-x}{b-a}a$ and $\phi(r) = \frac{2-(\varepsilon r)^2}{(1+(\varepsilon r)^2)^4}$.

and simplifying with the aid MATLAB computer software we have the result of the method compared with the exact solution in Table 2.

Table 2: The Projection Method for Example 2 with the Linear Generalized Inverse Multiquadric and Quadratic Generalized Inverse Multiquadric with Absolute Errors

i	x_i	Solution with Linear Generalized Inverse Multiquadric	Exact solution	Absolute error	Solution with Quadratic Generalized Inverse Multiquadric	Absolute error
1	0.000000	0.999990	1.000000	4.566557×10^{-8}	1.000000	3.716055×10^{-9}
2	0.055556	1.057127	1.057128	2.503576×10^{-7}	1.057128	1.213365×10^{-7}
3	0.111111	1.117520	1.117519	5.221358×10^{-7}	1.117519	3.272218×10^{-7}
4	0.166667	1.181360	1.181360	2.810283×10^{-7}	1.181359	1.320731×10^{-6}
5	0.222222	1.248849	1.248849	1.075667×10^{-7}	1.248848	6.922862×10^{-7}
6	0.277778	1.320194	1.320193	9.751285×10^{-7}	1.320192	6.721333×10^{-7}
7	0.333333	1.395613	1.395612	8.409910×10^{-7}	1.395611	8.738628×10^{-7}
8	0.388889	1.475341	1.475341	7.838330×10^{-7}	1.475339	1.316403×10^{-6}
9	0.444444	1.559624	1.559623	8.872456×10^{-7}	1.559622	1.797235×10^{-6}
10	0.500000	1.648722	1.648721	8.230098×10^{-7}	1.648719	1.936996×10^{-6}
11	0.555556	1.742910	1.742909	1.130465×10^{-6}	1.742907	2.487153×10^{-6}
12	0.611111	1.842479	1.842477	1.121861×10^{-6}	1.842476	1.945132×10^{-6}
13	0.666667	1.947735	1.947734	1.225780×10^{-6}	1.947731	2.654696×10^{-6}
14	0.722222	2.059005	2.059003	1.160124×10^{-6}	2.059000	3.359561×10^{-6}
15	0.777778	2.176632	2.176630	1.677757×10^{-6}	2.176628	2.399399×10^{-6}
16	0.833333	2.300977	2.300976	8.971588×10^{-7}	2.300971	4.461830×10^{-6}
17	0.888889	2.432427	2.432425	1.699060×10^{-6}	2.432422	3.554353×10^{-6}
18	0.944444	2.571386	2.571384	1.961140×10^{-6}	2.571381	3.768764×10^{-6}
19	1.000000	2.718283	2.718282	1.506753×10^{-6}	2.718277	4.644195×10^{-6}

Discussion

Numerical experiments using the quadrature-based projection methods shows the usefulness of our method. The methods were developed using $\phi(r) = \frac{2-(\varepsilon r)^2}{(1+(\varepsilon r)^2)^4}$ (linear generalized inverse multiquadric function) and $\phi(r) = \frac{3-6(\varepsilon r)^2+(\varepsilon r)^4}{(1+(\varepsilon r)^2)^6}$ (quadratic generalized linear multiquadric function) as the basis functions.

Table 1 compares the solution of the linear first order Volterra integro-differential equation with constructed method using linear generalized inverse multiquadric function and the Quadratic generalized multiquadric function with the absolute errors been 9.682569×10^{-6} as the maximum for the linear generalized inverse multiquadric function as the basis function while the solution with quadratic generalized multiquadric function has the maximum absolute errors as 2.876289×10^{-5}. These methods based on both basis functions are quite accurate but the generalized linear multiquadric function yields superior results for the first order Volterra integro-differential equation.

Table 2 compares the solution of the linear second order Volterra integro-differential equation with the constructed method using linear generalized inverse multiquadric function and the Quadratic generalized multiquadric function with the absolute errors been 1.961140×10^{-6} as the maximum using the linear generalized inverse multiquadric function as basis function while the solution with quadratic generalized linear multiquadric function as basis function has its absolute errors been 4.644195×10^{-6} as the maximum.

Conclusion

From the above methods, the quadratic generalized multiquadric function is a more accurate method for the solution of both first order and nth order Volterra integro-differential equations.

Declaration of conflicting interests

The author declared no potential conflicts of interest

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